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Invariants for the Critical Points in Network Models of Flow in Porous Media

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The use of "dimensional invariants" relating critical size fractions, coordination number, and dimension is investigated for fluid "invasion percolation" on correlated and uncorrelated networks. A new descriptor, the fraction of passable pores, is introduced to unify the treatment of drainage and imbibition (or bond and site percolation) processes, and to calculate new, approximate dimensional invariants. Not only do drainage and imbibition processes in lattices where pore and throat sizes are correlated have similar critical values for the fraction of passable pores for a given coordination number and dimension, but this fraction is also only slightly dependent on coordination number, namely to a power of about 0.17.

KEY WORDS: Critical points; dimensional invariants; percolation in 3D lattices; correlated lattices; immiscible fluid displacement in porous media.

1. INTRODUCTION

Percolation processes are either pore (site) or throat (bond) controlled and it is hence customary to determine critical points in terms of fractions of these elements. Except for rather standard situations, these critical fractions, e.g., the fraction of invadable pores, must be determined by carrying out a large number of simulations on appropriate grids. The existence of invariant expressions can circumvent this if certain simple properties, such as coordination number, are known. The current method of approximately determining critical points utilizes invariants based on statistical quantities and supplements previous work wherein we showed the wide applicability of statistical methods to obtain quantities important to two-phase flow in porous media.[1]

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We wish to communicate numerically derived invariant expressions relating critical fractions, coordination number, and dimension, at the invader breakthrough point (first critical point) as well as at the defender disconnection point (second critical point) when drainage and imbibition problems are modeled as invasion percolation processes. We shall also argue that a new descriptor introduced below, the fraction of passable pores, is an appropriate and extremely powerful tool for analyzing critical points and for producing new invariants more widely applicable than the former versions.

While the following methods are inspired by fluid flow in porous media, they are applicable to ordinary percolation, too. Indeed, ordinary percolation and invasion percolation will have identical first critical points, whether they are defined in the manner of Wilkinson and Willemsen,[2] or more loosely as a method of percolation where only those invadable bonds and sites are invaded which are adjacent to elements already containing the new phase. Only certain fractal properties may change.[3]

1.1. Representation of Results

1.1.1. Old Invariants. For the standard bond and site percolation problems there exist already simple "invariant" relationships for determining the critical breakthrough point in terms of "invadable" pore or throat fractions of a random network. The details of the geometry of the percolating network play no role. The relationship obtained by Vyssotsky *et al.*[4] for drainage (the bond problem) is particularly easy to use. With Z for the coordination number, t_{c1} for the fraction of throats larger than the first critical size, and E, the Euclidean dimension, the invariant is

$$Zt_{c1} = E/(E-1) \tag{1}$$

The product Zt_{c1} is the average number of invadable throats per pore. This expression gives the correct value of t_{c1} to within a few percent for a network of randomly distributed throat sizes.[5] The applicability of an extension of this relation to networks in which throats are not random but depend on neighboring pore sizes will form a part of this report.

For imbibition (the site problem) Scher and Zallen[6] state that if all pores are considered to be touching spheres of equal size, then at the first critical point the volume fraction of those considered penetrable is dependent on dimension only. The volume fraction or density ρ of "conducting" spheres is given by

$$f(Z)\, p_{c1} = \rho(E) \tag{2}$$

where f is the filling factor (or volume ratio between total sphere volume and sample volume). The value $f(Z)$ depends on the coordination number Z of the underlying lattice. For a honeycomb lattice $f = 0.6046$; for a square lattice, $f = 0.7854$; for a triangular lattice, $f = 0.9069$; and for a cubic lattice, $f = 0.5230$. The critical fraction of conducting spheres is p_{c1} and is called the fraction of invadable pores in our case. The dimensional parameter $\rho(E)$ is 1 in one dimension, 0.44 in two dimensions, and 0.15 in three dimensions. Rather than extend this relation, we intend to show that imbibition can be described quite satisfactorily using invariants which are structurally the same as those for drainage.

1.1.2. Extensions and New Invariants. We will extend the application of a Vyssotsky type of invariant to imbibition and to drainage in correlated networks (by which we mean networks wherein the size of neighboring elements is determined, in part, by an analytic relation). These invariants are to be presented in the form

$$Z^{\text{exponent}}\, t_{c1} = \text{const} \tag{3a}$$

$$Z^{\text{exponent}}\, p_{c1} = \text{const} \tag{3b}$$

The subscript $c1$ indicates values at the first critical point. The applicability of this extended Vyssotsky relation to the second critical point for both drainage and imbibition in uncorrelated and correlated lattices will be investigated. These invariants are to be presented in the form

$$Z^{\text{exponent}}\, t_{c2} = \text{const} \tag{4a}$$

$$Z^{\text{exponent}}\, p_{c2} = \text{const} \tag{4b}$$

where the subscript $c2$ indicates values at the second critical point where the defender becomes disconnected from the source.

Since our interest centers on flow in porous media, we propose a descriptive parameter which emphasizes "flow or no flow" and suggest the use of fractions of passable, or impassable, pores to characterize both drainage and imbibition processes. The term "passable" denotes pores surrounded by at least two open throats; the fraction of impassable pores is the complement of passable pores. We can reformulate the question from, "What is the fraction of invadable pores or throats at the critical point?," to, "What is the fraction of pores which can be passed through at a critical point?." It will turn out that this newly introduced descriptor permits one to unify not only imbibition processes with drainage processes, but drainage processes in lattices with both random and correlated structures. This recognizes that ultimately formulas such as those of Vyssotsky *et al.*[4]

and of Scher and Zallen[6] are based on arguments that turn on the average number of open throats per pore at the critical point. The new invariants may thus be viewed as an extension of a viewpoint which puts emphasis on the average number of next neighbor pores reachable from a given pore at the critical point. Such a viewpoint is indeed capable of considerable unification of results for different lattices and even different processes. This second type of invariant will be presented in the form

$$Z^{\text{exponent}} f^{P}_{\text{impass}} = \text{const} \tag{5}$$

or

$$Z^{\text{exponent}} f^{P}_{\text{pass}} = \text{const} \tag{6}$$

Equation (5) will apply to the first critical point, Eq. (6) to the second critical point. In all of the relations (3)–(6) the value of the exponent of Z is adjusted such that, on using the relevant experimental values ($t_{c1}, f^{P}_{\text{impass}}$, etc.), the product so formed, "the constant," is stable over the range of coordination numbers employed.

1.2. Methods

The details of our procedure can be found elsewhere[1,7,8] and we present here only sufficient information to carry the argument.

1.2.1. The Lattices. Lattices are created in which pores (sites) are selected from a distribution uniform between 0 and 1. Throats (bonds) are selected at random from a uniform distribution (uncorrelated case) or are selected on the basis of the size of adjacent pores such that the throats will be smaller than the pores (correlated case; specific examples will appear later).[8] The number of throats connecting a given pore to its neighbors is the coordination number Z; for two dimensions $2 \leqslant Z \leqslant 6$ and for three dimensions $2 \leqslant Z \leqslant 8$. The lattices are initially square in two dimensions or cubic in three dimensions except for the highest coordination numbers, where the lattices are triangular and octagonal in two and three dimensions, respectively. In two dimensions, low-coordination-number lattices are produced by randomly blocking throats in square lattices; in three dimensions, throats in cubic lattices are blocked. Pores and throats contained in unreachable domains which are a result of randomly blocking throats are not counted (that fraction is quite small for large lattices except for Z approaching 2). The simulations show no significant difference between using regular lattices of low coordination number and using a high-coordination-number lattice in which the coordination number has been lowered by some random blocking of throats. For very low coordination numbers, growing domains of the lattice are cut off. The difference between intended and effective coordination number grows quickly, and the dimensional invariants derived later no longer fit the data closely.

In two dimensions experimental values represent averages from 50 lattices of size 128×256; in three dimensions, 20 lattices of size $48 \times 48 \times 96$ were used unless a *caveat* is introduced. The breakthrough points are determined experimentally by simply stepping toward them in steps of 0.01 of the controlling pore or throat size. The corresponding fractions of invadable and penetrable pores can be determined either from the simulations or analytically (the agreement between calculated and experimental fractions of passable pores is extremely good and deteriorates only for very low coordination numbers[1,7]).

1.2.2. The Percolation Process. In the initial state the lattice is fully occupied by a "defender fluid"; the source of the "invader fluid" is one edge of the grid and the sink is the opposite edge. Advance of invader is controlled by a simple size rule; in drainage the size of the throat and in imbibition the size of the pore are checked. The physical nature of displacement is maintained by ensuring connectivity of fluid to sink and/or source. The earliest examples are given by Fatt[9] and deGennes and Guyon.[10] These early investigators also made a natural connection to so-called capillary pressure curves, which display the pressure as a function of the amount of fluid pressed into a porous medium. The possibly better known work by Broadbent and Hammersley[11] lists fluid percolation only as an example of a percolation experiment without entering into any details. Invasion percolation as introduced by Wilkinson and Willemson[2] is a special case insofar as only one bond or site is invaded at the time, namely the "most invadable" element. We are interested only in approximate critical values and hence invade all invadable elements (pores or throats) which can be reached by invader each time the "pressure" is changed by a finite step. Then the connectivity of the defender is checked, since we also wish to detect the second critical point.

The important difference between the current and the cited work consists not in the method of percolation, but rather in the use of correlation of throats to pore sizes along the lines introduced by Li *et al.*[8] The fact that invading fluid has to pass through pores in order to invade a lattice suggested the introduction of the concept of passable pores (sites), which considerably unifies numerical results and thus shows up underlying similarities between processes in different lattices and even different processes. It is the power of this notion we will examine next through numerical examples.

2. BREAKTHROUGH FOR PERCOLATION MODELS OF FLOW IN POROUS MEDIA

2.1. Drainage for an Uncorrelated Network

To illustrate our data and our procedure, we show, in Table I, the results for uncorrelated networks in both two and three dimensions. We present the primitive data, the coordination number Z and fraction of invadable throats t_{c1}, in the first two columns. Some of the critical fractions t_{c1} are of course known in the literature and may even be exact; we quote, for comparison, the following values for t_{c1} in two dimensions: 0.347 for $Z=6$, 0.500 for $Z=4$, and 0.653 for $Z=3$, which is for a honeycomb lattice.[5,12] The value we obtain for t_{c1} for $Z=6$ is 0.355, and for $Z=4$, $t_{c1}=0.508$, the small deviations being a reflection of the finite size of our lattice and the step size of 0.01. We list the Vyssotsky product Zt_{c1} (i.e., the average number of invadable throats per pore) in the third column, and the new invariant $Z^0 f^P_{\text{impass}}$ in the last column.

Except for values of Z close to the limits ($Z=2$ and $Z=8$), both the Vyssotsky product Zt_{c1} and the new invariant $Z^0 f^P_{\text{impass}}$ are indeed well behaved. The former has the values 2.02 and 1.52, very close to the values of 2.00 and 1.50 predicted by Eq. (1). The new invariant is just the fraction of impassable pores f^P_{impass} (since the best value of the exponent of Z is zero) and this indeed is essentially constant for all Z values. One has $f^P_{\text{impass}}=0.30$ for the two-dimensional case and $f^P_{\text{impass}}=0.52$ for the three-dimensional case.

The instability in the invariants at the largest coordination number is occasioned by a change in lattice type and at the lowest value of Z by the

Table I. Drainage in Uncorrelated Networks

Two dimensions				Three dimensions			
Z	t_{c1}	Zt_{c1}	$Z^0 f^P_{\text{impass}}$	Z	t_{c1}	Zt_{c1}	$Z^0 f^P_{\text{impass}}$
2.1	0.963	2.02	0.27	2.0	0.763	1.52	0.48
2.25	0.902	2.03	0.28	2.5	0.607	1.53	0.51
2.5	0.807	2.02	0.29	3.0	0.506	1.52	0.52
3.0	0.680	2.04	0.29	3.5	0.435	1.52	0.52
3.5	0.581	2.03	0.30	4.0	0.381	1.52	0.53
4.0	0.508	2.03	0.30	5.0	0.306	1.53	0.52
6	0.355	2.13	0.31	6.0	0.257	1.54	0.52
				8.0	0.186	1.49	0.55

fact that the desired coordination number may differ significantly from the effective coordination number because of the "cutoff" regions referred to earlier.

2.2. Drainage in Lattices with Correlated Throat Sizes

Since for many porous media, such as Indiana limestone, the pore and throat sizes appear to be strongly correlated,[13] it is of interest to see whether the notion of invariants of the types investigated in the previous section can be applied to percolation processes on correlated lattices. We investigated two different but illustrative correlations; in the first, the throat is taken to have the same size as the smaller of the two pores it connects, and in the second the throat size is taken to be given by the product of the normalized size of the adjacent pores.

Table II shows breakthrough critical sizes in drainage for these correlated lattices for a series of coordination numbers. As before, the generalized Vyssotsky invariant is quite stable within a given lattice type, but again is not stable when the lattice type is changed. Nor is the generalized Vyssotsky relation stable to a change in correlation type. The values for the Vyssotsky product for the two differently correlated three-dimensional lattices are fitted best by, respectively, $Z^{1.20}t_{c1} = 1.5$ and $Z^{1.55}\, t_{c1} = 1.9$ for the range $2 \leqslant Z \leqslant 6$.

The new invariant, $Z^{\text{exponent}} f^P_{\text{impass}}$, appears to be stable within a given lattice type and is also stable with respect to changes in the underlying geometry. Moreover, it is virtually the same for differently correlated lattices and can be encompassed by the form

$$Z^{-0.17} f^P_{\text{impass}} = 0.59$$

Table II. Drainage in Three-Dimensional Correlated Lattices

	Throat size T_{ij} = min of P_i or P_j			Throat size T_{ij} = product $P_i \times P_j$		
Z	t_{c1}	$Z^{1.20}t_{c1}$	$Z^{-0.17} f^P_{\text{impass}}$	t_{c1}	$Z^{1.55}t_{c1}$	$Z^{-0.17} f^P_{\text{impass}}$
2.0	0.678	1.56	0.51	0.659	1.93	0.49
2.5	0.502	1.51	0.56	0.471	1.95	0.53
3.0	0.394	1.47	0.58	0.352	1.93	0.56
3.5	0.326	1.47	0.59	0.272	1.90	0.57
4.0	0.281	1.48	0.59	0.219	1.88	0.58
5.0	0.215	1.48	0.60	0.156	1.90	0.58
6.0	0.180	1.55	0.59	0.118	1.91	0.58
8.0	0.064	0.77	0.59	0.071	1.77	0.58

Not only is f^P_{impass} the same for differently correlated networks for a given Z, but the very small exponent indicates also that f^P_{impass} is almost coordination number independent. One should, however, notice that the effect of throat correlation is destroyed by lower coordination number, since, as is clear from the tables, t_{c1} must approach unity as the intended coordination number Z approaches 1.5 (the "effective" coordination number determined for the domain of accessible pores converges to two).

2.3. Imbibition

The case of imbibition is of course different from the previously discussed cases in that it is a pore-controlled process. Nevertheless, expressions can be derived which are analogous to those for drainage.[1,7] The values for the controlling fraction p_{c1}, the extended Vyssotsky relation, and the new passable pores invariant are all listed in Table III. The fractions of invadable pores (pores small enough to be invaded) follow the equation

$$Z^{0.80} p_{c1} = 1.35$$

The fraction of impassable pores, which is now defined as the fraction of invadable pores adjacent to at most one invadable pore at a given pressure, follows the relation

$$Z^{-0.17} f^P_{\text{impass}} = 0.58$$

Notice again the very small exponent in the three-dimensional case, which indicates that f^P_{impass} is almost coordination number independent. Moreover, this expression for imbibition is the same as obtained in the previous section on drainage for the two correlated cases! This makes the last formula very widely applicable indeed.

Table III. Imbibition

Z	p_{c1}	$Z^{0.80}p_{c1}$	$Z^{-0.17}f^P_{\text{impass}}$
2.0	0.788	1.37	0.51
2.5	0.650	1.35	0.56
3.0	0.555	1.34	0.58
3.5	0.494	1.34	0.59
4.0	0.443	1.34	0.59
5.0	0.369	1.34	0.59
6.0	0.322	1.35	0.59
8.0	0.250	1.32	0.59

3. THE ENDPOINT IN NETWORK MODELS OF FLOW IN POROUS MEDIA

3.1. Representation of Results

The investigation of the previous section has indicated the existence of two different invariants characterizing breakthrough. Perhaps more important, it has begun to appear that the notion of "passable pores" has the potential to unify a number of different processes via the relation

$$Z^{\text{exponent}} f^{P}_{\text{impass}} = \text{const}$$

This potential has persuaded us to apply it to the second critical point, the point of disconnection of the defender fluid from the source. After all, it is this point which determines the amount of fluid which may be displaced from a porous medium.

The fractions which best describe breakup of the defender are not quite the same as those useful to determine breakthough. While breakthrough is generally well characterized by relations involving either the fraction of invadable throats (pores in imbibition) or the fraction of impassable pores, the second critical point is best described by relations involving the complements of the latter fractions.

3.2. Results

In this section we report the values, at disconnection of defender, of a set of characteristic quantities beginning with the usual controlling element t_{c2}, the fraction of throats greater than a critical size, followed by the Vyssotsky-like invariant $Z^{\text{exponent}}\, t_{c2}$, and the new invariant $Z^{\text{exponent}} f^{P}_{\text{pass}}$. The data were obtained in the same manner as those for the breakthrough points; Table IV is for drainage in an uncorrelated lattice, Table V is for

Table IV. Disconnection: Drainage in an Uncorrelated Three-Dimensional Lattice

Z	t_{c2}	$Z^{-1/2} t_{c2}$	$Z^{-0.17} f^{P}_{\text{pass}}$
3	0.563	0.25	0.46
3.5	0.495	0.27	0.45
4.0	0.447	0.28	0.46
4.5	0.408	0.28	0.45
5.0	0.373	0.28	0.46
6	0.320	0.28	0.46
8	0.254	0.26	0.45

Table V. Disconnection: Drainage in Two Correlated Three Dimensional Lattices

	Throat size $T_{ij} = \min$ of P_i or P_j			Throat size $T_{ij} =$ product $P_i \times P_j$		
Z	t_{c2}	$Z^{-0.10}t_{c2}$	$Z^{-0.45}f^P_{\text{pass}}$	t_{c2}	$Z^{-0.10}t_{c2}$	$Z^{-0.45}f^P_{\text{pass}}$
3.0	0.445	0.50	0.27	0.469	0.48	0.27
3.5	0.415	0.52	0.27	0.423	0.51	0.27
4.0	0.408	0.52	0.28	0.398	0.52	0.26
4.5	0.429	0.49	0.30	0.406	0.51	0.28
5.0	0.426	0.49	0.29	0.406	0.51	0.28
6	0.470	0.44	0.30	0.408	0.50	0.28
8	0.576	0.34	0.30	0.469	0.43	0.28

drainage in the two kinds of correlated lattices, and Table VI is for imbibition. These results show the same kinds of connections found while investigating breakthrough points, the principal difference being the replacement of f^P_{impass} by f^P_{pass}. Once again, drainage in correlated lattices and imbibition have their endpoints at the same fractions, this time of passable pores, the resulting relation being

$$Z^{-0.45} f^P_{\text{pass}} = 0.3$$

For drainage in uncorrelated lattices the relation is

$$Z^{-0.17} f^P_{\text{pass}} = 0.46$$

The value of 0.17 is the same as that which was found for imbibition (and drainage in correlated lattices) at the breakthrough point. The constant, too, is roughly the same, about three-quarters of the previously found value.

Table VI. Disconnection: Imbibition in a Three-Dimensional Lattice

Z	t_{c2}	$Z^{-0.10}t_{c2}$	$Z^{-0.45}f^P_{\text{pass}}$
3.0	0.735	0.24	0.32
3.5	0.697	0.27	0.28
4.0	0.678	0.28	0.31
4.5	0.670	0.28	0.30
5.0	0.670	0.28	0.30
6	0.693	0.26	0.31
8	0.698	0.25	0.27

4. DISCUSSION

Table VII summarizes the exponents and constants found for the invariants investigated in this work. The column headings A, B, C, and D indicate results for different processes: (A) for drainage in uncorrelated networks, (B) for drainage in minimally correlated networks, (C) for drainage in product-correlated networks, and (D) for an imbibition process. For the "new" section of Table VII, the row labels "exponent" and "constant" indicate exponents and constants for our new invariant in the form

$$Z^{\text{exponent}} f^{P}_{\text{impass}} = \text{const} \qquad \text{and} \qquad Z^{\text{exponent}} f^{P}_{\text{pass}} = \text{const}$$

at the breakthrough and disconnection critical points, respectively. For the "Vyssotsky" section of Table VII, the row labels indicate the exponent and constant for the Vyssotsky relations

$$Z^{\text{exponent}}\, t_{c1} = \text{const} \qquad \text{and} \qquad Z^{\text{exponent}}\, t_{c2} = \text{const}$$

for breakthrough and disconnection critical points, respectively, of a drainage process. Since, in the Vyssotsky case, the invariant for imbibition is expressed in terms of pores, the values quoted in column D of the Vyssotsky section of Table VII are for the exponent and constant for the form

$$Z^{\text{exponent}}\, p_{c1} = \text{const} \qquad \text{and} \qquad Z^{\text{exponent}}\, p_{c2} = \text{const}$$

It is clear from the table that, for a given coordination number, the fractions of impassable pores at breakthrough are effectively the same for the two types of correlation and for imbibition, since the invariants are identical! The extended Vyssotsky invariant are much less stable.

The equivalence between drainage in a minimally correlated lattice

Table VII. Values of the Exponent and Constant for the Invariants[a]

		Invader Breakthrough				Defender disconnection			
Relation	Values	A	B	C	D	A	B	C	D
New	Exponent	0	−0.17	−0.17	−0.17	−0.17	−0.45	−0.45	−0.45
	Constant	0.5	0.58	0.58	0.58	0.46	0.29	0.29	0.29
Vyssotsky	Exponent	1	1.20	1.55	0.80	−0.5	−0.10	−0.10	−0.10
	Constant	1.5	1.5	1.9	1.32	0.27	0.5	0.5	0.25

[a] See text for explanation of headings A–D.

and imbibition is relatively easily established. It is true that correlation permits one to relate throat dependence to pore sizes so that the pores become the controlling elements for drainage as well as for imbibition. Since, by definition, throats are supposed to be smaller than pores, the receding menisci in drainage are stopped by the small throats, while the advancing menisci in imbibition are stopped by the large pores. For drainage in a correlated network created by the relation $t = \min(p_1, p_2)$, percolation is established by seeking the larger throats. The boundaries of the percolating structure are established by the smaller throats. For imbibition one can create a percolating network by first replacing the pore sizes in the original network by their complement $1 - p$; i.e., the large pores are replaced by the small, and of course, since the pores are uncorrelated, this is still an uncorrelated network.

One can now attempt to establish percolation by selecting the smaller pores. Now the boundaries of the percolating structure are set by the larger pores. The percolating structure is the same in both cases! Consequently, for a given Z the fraction of impassable pores f^P_{impass} at breakthrough of invader in the correlated network $t = \min(p_1, p_2)$ is the same as for imbibition on an uncorrelated network.

The case where $t = (p_1 * p_2)$ is more complicated. Suppose then that we have a pore in a two-dimensional lattice, surrounded by four neighboring pores which shall be smaller than the central pore. If we now multiply the throat sizes in this local system by the size of the central pore, then nothing happens to the ratio of the sizes of these throats. If only half of the neighboring pores are smaller than the central pore, then multiplication of the throats with the larger pore size once again will not disturb the order of the throats except for differentiating those which by the previous method of correlation were the same. Hence there is no local change if we change the method of correlation, although this will not apply to the whole lattice in this strict form. One should, however, expect a close, but not perfect, correspondence between the percolating structures for the correlation $t = \min(p_1, p_2)$ and the correlation $t = (p_1 * p_2)$. Consequently, the close correspondence between the values of f^P_{impass} for the two correlated networks shown in Table II for breakthrough and in Table V for disconnection is not surprising. It can be shown that we cannot change from one method of correlation to another without changing in some degree the number of invadable throats per pore.[1,7]

The similarity in the invariant $Z^{-0.17} f^P_{\text{pass}}$ for the drainage disconnection point for an uncorrelated network and the invariant $Z^{-0.17} f^P_{\text{impass}}$ for breakthrough in imbibition is remarkable, but is left without explanation except to say that dependence on uncorrelated throats in the former appears to be the same as for uncorrelated pores in the latter.

5. CONCLUDING REMARKS

Our investigations show that a generalization of the Vyssotsky dimensional invariant, $Z^{\text{exponent}} t_c = \text{const}$ or $Z^{\text{exponent}} p_c = \text{const}$, can successfully describe both breakthrough of invader fluid and disconnection of defender fluid in correlated lattices. Exponent and constant depend on the case under investigation. To remove this sensitivity to the details of the case, we have introduced the new parameters f^P_{impass} and f^P_{pass} to characterize critical points. For a given coordination number these latter quantities are the same for drainage on our correlated lattices and for imbibition. They also allow the introduction of numerically quite stable invariants of the form $Z^{\text{exponent}} f^P_{\text{impass}} = \text{const}$ or $Z^{\text{exponent}} f^P_{\text{pass}} = \text{const}$. As is evident from Table VII, these new, numerically derived invariants are more "portable" than those of the "extended Vyssotsky invariant" type.

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